Eigenvalue Similarity Rules for Symmetric Cross-Ply Laminated Plates

E. J. Brunelle*
U. S. Air Force Institute of Technology, Dayton, Ohio

As recently demonstrated, the stability and vibration problems of orthotropic plates can be efficaciously treated in suitable affine spaces and similarity rules independent of individual problem boundary conditions can be formulated and very effectively used in analysis *and* design. The present work extends these concepts to symmetric cross-ply laminated plates by employing the affine stretching process, utilizing some previous material constant definitions due to Tsai, and generalizing the existing similarity rules. As examples, similarity rules are employed to find the frequency spectrum of a cross-ply symmetric plate from the corresponding isotropic solution and a similarity solution is presented for the vibration of an axially loaded simply supported cross-ply symmetric plate. Buckling similarity rules are also presented.

Nomenclature

a,b	= plate physical dimensions
a_0,b_0	= plate affine space dimensions
b_1,b_2	= cross-ply symmetric plate parameters
D_{ij}	= plate constants
D	= differential operator
D^*	= generalized rigidity for a specially orthotropic layer
D_c^*	= generalized rigidity for a cross-ply symmetric laminated plate
E_{11},E_{22},G_{12}	= orthotropic slab constants
k_{x_0},k_{y_0}	= plate affine buckling coefficients
m,n	= number of half waves in x_0 and y_0
$\bar{m}$	= mass per unit middle surface
M_y	= bending moment per unit width
N_x,N_y	= in-plane loads per unit width
Q_{ij}	= modulus components
V_y	= effective shear per unit width
w	= lateral plate displacement
$W_{mn}(y_0)$	= plate mode shape
x,y	= plate physical coordinates
x_0,y_0	= plate affine space coordinates
$\bar{\alpha}$	= similarity parameter
$\alpha, i\beta$	= roots of operator equation
ϵ	= generalized Poisson's ratio
ν_{12}	= principal Poisson's ratio
Π_1,Π_2	= differential equation coefficients used in similarity rules
Υ	= similarity parameter
ω_{mn}	= physical frequency
Ω_{mn_0}	= affine frequency

Equation of Motion and Its Affine Counterparts

The equation of motion of a biaxially loaded, freely vibrating, rectangular cross-ply (symmetric) laminated plate is given by

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}-N_x\frac{\partial^2 w}{\partial x^2}-N_y\frac{\partial^2 w}{\partial y^2}+\bar{m}\ddot{w}=0 \tag{1}$$

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*Professor, Department of Aeronautics and Astronautics (presently with Rensselaer Polytechnic Institute, Troy, NY). Member AIAA.

Introducing the affine transformations[1,2] $x=(D_{11})^{1/4}x_0$ and $y=(D_{22})^{1/4}y_0$, Eq. (1) is recast in the following form [note that $a=(D_{11})^{1/4}a_0$ and $b=(D_{22})^{1/4}b_0$]:

$$\frac{\partial^4 w}{\partial x_0^4}+2D_c^*\frac{\partial^4 w}{\partial x_0^2\partial y_0^2}+\frac{\partial^4 w}{\partial y_0^4}+\left(\frac{\pi}{b_0}\right)^2 k_{x_0}\frac{\partial^2 w}{\partial x_0^2}+\left(\frac{\pi}{a_0}\right)^2 k_{y_0}\frac{\partial^2 w}{\partial y_0^2}+\bar{m}\ddot{w}=0 \tag{2}$$

where

$$k_{x_0}=-N_x b^2/\pi^2\sqrt{D_{11}D_{22}}$$

$$k_{y_0}=-N_y a^2/\pi^2\sqrt{D_{11}D_{22}}$$

$$D_c^*=(D_{12}+2D_{66})/\sqrt{D_{11}D_{22}}$$

D_c^* is less than the specially orthotropic layer value of D^*, as shown in the next section. The reader is reminded that the range of D^* is the closed interval zero to one,[3,4] i.e., $0\le D^*\le 1$.

For a single-layer, specially orthotropic plate, the D_{ij} terms are simply related to the material constants E_{11}, E_{22}, G_{12}, and ν_{12}. The values of the D_{ij} terms for the cross-ply (symmetric) laminated plate were presented by Tsai[5] and are used in the next section to develop a method for calculating D_c^* and the above used constants D_{11}, D_{22}, and their geometric mean $\sqrt{D_{11}D_{22}}$.

Values of D_c^* and Associated Terms for Cross-Ply Symmetric Laminated Plates

Following Tsai,[5]

$$D_{11}=b_1Q_{11}(t^3/12) \tag{3}$$

$$D_{22}=b_2Q_{11}(t^3/12) \tag{4}$$

where

$$b_1=(F-1)P+1$$

$$b_2=F-(F-1)P=(F+1)-b_1$$

$$F=Q_{22}/Q_{11}=E_{22}/E_{11}$$

$$P=\frac{1+[M(N-1)+2(N+1)]M(N-3)/(N^2-1)}{(1+M)^3}$$

and N is the number of layers (odd) and M the ratio of the total thickness of 0° plies to the total thickness of 90° plies.

Using the results of $Q_{11} = Q_{22}/F$, the geometric mean of D_{11} and D_{22} is found to be

$$\sqrt{D_{11}D_{22}} = \sqrt{b_1 b_2 / F}(\sqrt{Q_{11}Q_{22}})t^3/12$$

and thus the desired value of D_c^* is

$$D_c^* = (F/b_1 b_2)^{1/2} Q^* \tag{5}$$

where Q^* is defined as

$$Q^* = (Q_{12} + 2Q_{66})/\sqrt{Q_{11}Q_{22}} \tag{6}$$

and is identically equal to D^* for a specially orthotropic single layer. The value of $(F/b_1 b_2)^{1/2}$ is always less than unity, as established by a careful numerical search; hence, the upper bound of D_c^* is less than the upper bound of D^* (namely, unity).

This is an important point, since it insures that the similarity rules to be presented are of the same form for *both* cross-ply symmetric plates *and* specially orthotropic plates.

Overview of Solution Process for Two Parallel Sides Simply Supported: Motivation for Seeking Similarity Rules

Letting the sides $x_0 = 0$, a_0 be simply supported allows a solution of the form

$$w(x_0, y_0, t) = W_{mn}(y_0)(\sin m\pi x_0/a_0)e^{i\omega_{mn}t} \tag{7}$$

to be passed through Eq. (2), yielding an ordinary differential equation in W_{mn} as

$$W_{mn}^{IV} - 2\left(\frac{m\pi}{a_0}\right)^2 \Pi_1 W_{mn}^{II} + \left(\frac{m\pi}{a_0}\right)^4 \Pi_2 W_{mn} = 0 \tag{8}$$

The roots of the corresponding operator equation are given by (the roots are of this form if $-\Pi_2 > 0$)

$$D = \pm\alpha, \pm i\beta \tag{9}$$

where†:

$$\alpha b_0 = (mb_0/a_0)\pi\sqrt{\Pi_1 + R} \tag{10}$$

$$\beta b_0 = (mb_0/a_0)\pi\sqrt{-\Pi_1 + R} \tag{11}$$

$$R = \sqrt{(\Pi_1)^2 - \Pi_2} \tag{12}$$

$$\Pi_1 = D_c^* - k_{y_0}/2m^2 \tag{13}$$

$$\Pi_2 = 1 - (a_0/mb_0)^2 k_{x_0} - (a_0/mb_0)^4 \Omega_{mn0}^2 \tag{14}$$

$$\Omega_{mn0}^2 = \bar{m}(b_0/\pi)^4 \omega_{mn}^2 \tag{15}$$

Thus, the general solution form for Eq. (8), under the assumption of Eq. (9), is given as

$$W_{mn}(y_0) = A_{mn}\sinh\alpha y_0 + B_{mn}\cosh\alpha y_0 + C_{mn}\sin\beta y_0 + D_{mn}\cos\beta y_0 \tag{16}$$

Choosing the appropriate boundary conditions (W and/or $\partial W/\partial y_0$ for geometric conditions and M_y and/or V_y for natural conditions), where M_y and V_y are given by

$$\frac{M_y}{(D_{22})^{1/2}} = \frac{\partial^2 w}{\partial y_0^2} + \epsilon D_c^* \frac{\partial^2 w}{\partial x_0^2}$$

and

$$\frac{-V_y}{(D_{22})^{1/4}} = \frac{\partial^3 w}{\partial y_0^3} + (2-\epsilon)D_c^* \frac{\partial^3 w}{\partial x_0^2 y_0}$$

the frequency spectrum can be found for any k_{x_0}, k_{y_0} combination that yields a stable (unbuckled) solution *by direct calculation*‡ [i.e., by using Eq. (16) and its derivatives to generate a transcendental equation that ensures satisfaction of the desired boundary conditions at $y_0 = 0, b_0$; the equation is of the general form $f(\alpha b_0, \beta b_0, \epsilon, D_c^*) = 0$, where αb_0 and βb_0 are defined by Eqs. (10) and (11)].

Only a few such problems have ever been done[6-9] *in detail*§ even for the *isotropic case*. Although the number of cases that need to be solved are greatly reduced by the introduction of D_c^* and the affine space concept, an easier method for obtaining solutions would be extremely helpful. In fact, such a method for obtaining solutions exists in the form of similarity rules. The idea of similarity rules (a recently coined term) for plate theory is not a new one (e.g., see the works of Wittrick[10] and Shulesko[11]), but it was not sufficiently developed into a really useful form until recently.[12] The next section discusses their application to the cross-ply symmetric laminated plate.

The Similarity Rules

Noticing that the arguments of Eq. (16) may be written as

$$\alpha y_0 \equiv \alpha b_0 (y_0/b_0)$$

and

$$\beta y_0 \equiv \beta b_0 (y_0/b_0)$$

it is seen that two solutions to Eq. (16) will be "similar" (the precise definition follows shortly) if the values of αb_0 and βb_0 are the same for both cases. Particularly note that *different combinations* of Ω, D_c^*, a_0/b_0, k_{x_0}, etc., *can yield the same values* of αb_0 and βb_0. Thus, from Eqs. (10-12) the similarity rules for two solutions (the solutions are labeled state 1 and state 2, which in the sequel will be shortened to just subscripts 1 and 2) are given by

$$\left\{\frac{mb_0}{a_0}\sqrt{\Pi_1}\right\}_{\text{state 1}} = \left\{\frac{mb_0}{a_0}\sqrt{\Pi_1}\right\}_{\text{state 2}} \tag{17}$$

$$\left\{1 - \frac{\Pi_2}{(\Pi_1)^2}\right\}_{\text{state 1}} = \left\{1 - \frac{\Pi_2}{(\Pi_1)^2}\right\}_{\text{state 2}} \tag{18}$$

Thus, if the solution corresponding to state 1 is known, the solution for *any generic* state 2 is known (or vice versa).

It is important to note that these similarity rules are *independent* of the boundary conditions (they would be *much less useful if they were to depend on boundary conditions*). These simple rules provide an elegant solution technique for saving copious amounts of money, labor, and time. The next section provides several simple applications of the similarity rule concept.

†For simplicity, the notation α, β is used to denote the subscripted quantities α_{mn}, β_{mn}.

‡Note that there exists a subcase in which $\omega_{mn} \equiv 0$; that is, the bi-eigenvalue buckling solution for minimum k_{x_0}, k_{y_0} pairs.

§The bulk of the solutions relate to only the fundamental frequency for the case of $k_{x_0} = k_{y_0} = 0$.

Specialized Examples of Similarity Rules

In this section, the subscript c will be implied, but deleted for simplicity.

Uniaxial Buckling

In this example, the time variable is suppressed (thus $\omega_{mn}=0$) and k_{y_0} is set to zero such that only axial loading in the x_0 direction is considered. The similarity rules simplify to the following:

$$\left(\frac{mb_0}{a_0}\right)_1^2 D_1^* = \left(\frac{mb_0}{a_0}\right)_2^2 D_2^* \tag{19}$$

$$\left[1-\left(\frac{a_0}{mb_0}\right)_1^2 (k_0)_1\right]\left(\frac{D_2^*}{D_1^*}\right)^2 = 1-\left(\frac{a_0}{mb_0}\right)_2^2 (k_0)_2 \tag{20}$$

where k_{x_0} has been shortened to k_0. Now, assuming that the m_1 branch of the minimum buckling solution is known for a particular plate *and* specified boundary¶ conditions [i.e., $(k_0)_1$ vs $(a_0/b_0)_1$ is known for a given m_1 and D_1^*], Eqs. (19) and (20) are rearranged so as to demonstrate the *algebraic* construction (*of all other branches, for all other D^* values*) of all related buckling curves. Therefore, upon selecting the desired branch m and the desired D^* value (m_2 and D_2^*), Eq. (19) yields,

$$\left(\frac{a_0}{b_0}\right)_2 = \frac{m_2}{m_1}\left(\frac{a_0}{b_0}\right)_1\left(\frac{D_2^*}{D_1^*}\right)^{1/2} \tag{21}$$

Thus, at this stage, the proper a_0/b_0 has been found for state 2 and the corresponding value of k_0 must be found from Eq. (20); after some rearrangement and substitution, the desired result is

$$(k_0)_2 = \frac{D_1^*}{D_2^*}\left(\frac{mb_0}{a_0}\right)_1^2\left\{\left[\left(\frac{a_0}{mb_0}\right)_1^2 (k_0)_1 - 1\right]\left(\frac{D_2^*}{D_1^*}\right)+1\right\} \tag{22}$$

Thus, picking any *one* branch of the uniaxial curves found in Ref. 13, any other branch for the same problem may be constructed solely by the direct algebraic manipulations given in Eqs. (21) and (22). This represents an enormous saving in time and labor over the straightforward method of solving *for each point on each curve* by computer-generated stability determinant solutions. This process is pictorially represented in Fig. 1, where from the prominently displayed state 1, all other curves may be found for any m_2 and any D_2^* (i.e., not just the values of D^* shown—all values!).

Free Vibration without In-Plane Loading

In this example, the in-plane loading is suppressed (thus $k_{x_0}=k_{y_0}=0$) such that the usual free-vibration problem remains. The similarity rules [Eqs. (17) and (18)] upon some rearrangement simplify to

$$\left(\frac{a_0}{b_0}\right)_2 = \frac{m_2}{m_1}\left(\frac{a_0}{b_0}\right)_1\left(\frac{D_2^*}{D_1^*}\right)^{1/2} \tag{23}$$

and

$$(\Omega_{mn0}^2)_2 = (\Omega_{mn0}^2)_1 + \left(\frac{mb_0}{a_0}\right)_1^4\left[\left(\frac{D_1^*}{D_2^*}\right)^2 - 1\right] \tag{24}$$

¶This is where the boundary conditions enter into the final problem (remember that the similarity rules were *independent* of the boundary conditions).

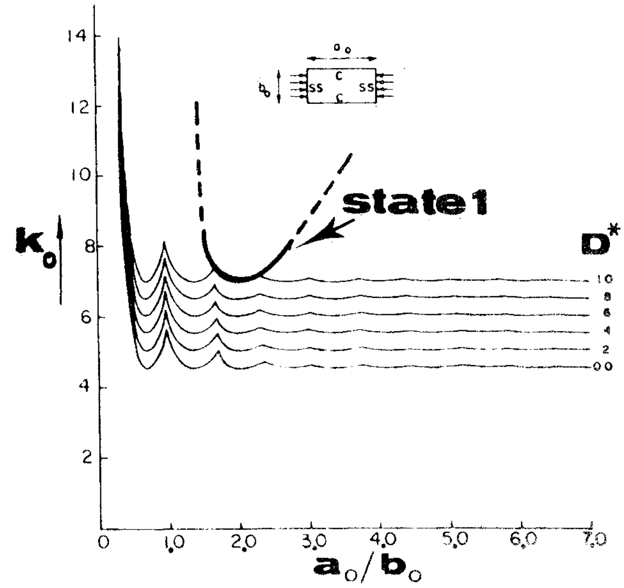

Fig. 1 Uniaxial buckling coefficients k_0 vs affine aspect ratio a_0/b_0 for SS-CL-SS-CL boundary conditions.

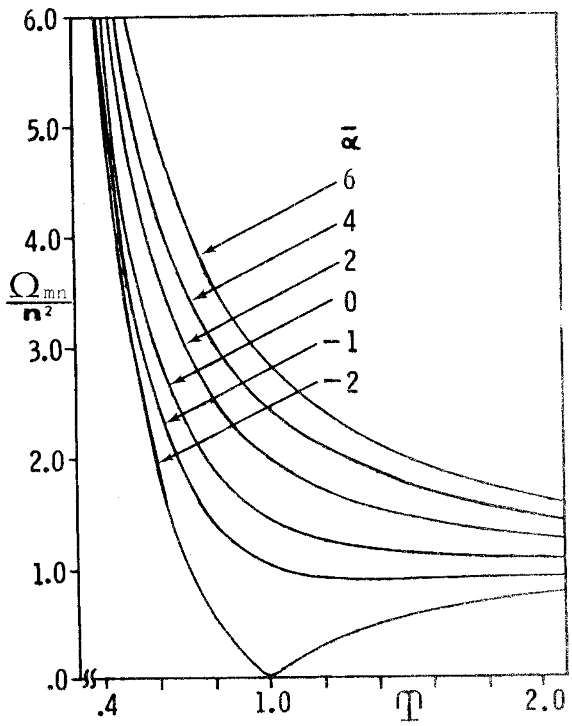

Fig. 2 Ω_{mn}/n^2 vs Υ with $\bar{\alpha}$ as a parameter for a SS-SS-SS-SS plate.

Now, if state 1 is for the $m=1$ *isotropic case* [$b_1=b_2=D_1^*=1$, $(a_0/b_0)_1=a/b$ and $(\Omega_{mn0})_1=(\Omega_{1n})_{\text{isotropic}}$], a very useful form of the similarity rules presents itself. Namely,

$$(a_0/b_0)_2 = m_2(a/b)(D_2^*)^{1/2} \tag{25}$$

$$\Omega_{m2n0}^2 = (\Omega_{1n}^2)_{\text{isotropic}} + (b/a)^4[(D_2^*)^{-2}-1] \tag{26}$$

Thus the frequency spectrum of *any* cross-ply symmetric plate is found from the corresponding isotropic plate frequencies ($\Omega_{11},\Omega_{12},\Omega_{13},\ldots$, vs a/b) just by carrying out the algebraic results of Eqs. (25) and (26). Once again, enormous savings are realized.

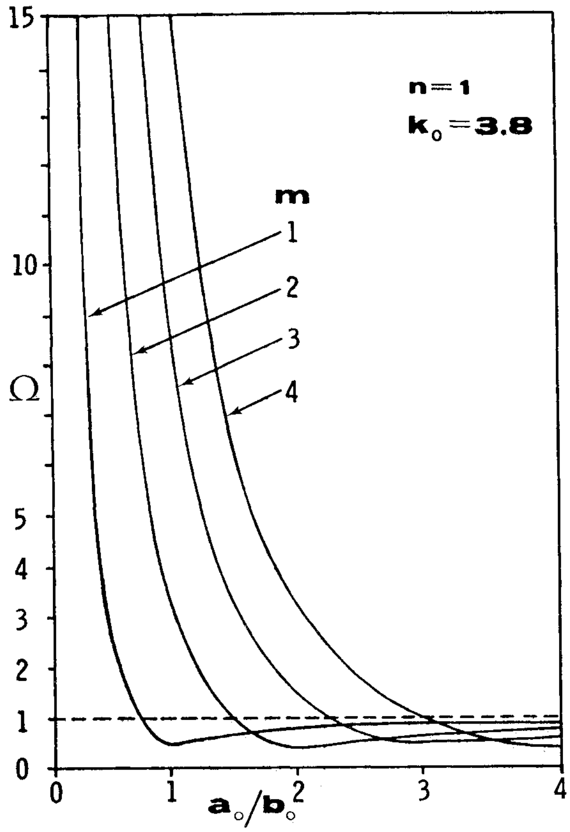

Fig. 3 **Ω vs a_0/b_0 with given values of m, n, and k_0 for a SS-SS-SS-SS plate.**

Freely Vibrating, Uniaxially Loaded Simply Supported Plate

For all four sides simply supported, a solution of the form

$$w(x_0,y_0,t) = W_{mn}(\sin m\pi x_0/a_0)(\sin n\pi y_0/b_0)e^{i\omega_{mn}t} \tag{27}$$

will pass through Eq. (2). Setting $k_{y_0} \equiv 0$ (for x_0 uniaxial loading) using the previous Ω_{mn} definition and defining

$$\bar{\alpha} = 2D_c^* - k_{x_0}/n^2 \tag{28}$$

and

$$\Upsilon = \{na_0/mb_0\} \tag{29}$$

where Υ is the similarity parameter [besides being the fundamental variable, its use is to relate the states 1 and 2 affine aspect ratios, namely $(a_0/b_0)_2 = (m/n)_2(n/m)_1(a_0/b_0)_1$], the similarity solution for all four sides simply supported is given by

$$\Omega_{mn}/n^2 = (1 + \bar{\alpha}\Upsilon^{-2} + \Upsilon^{-4})^{1/2} \tag{30}$$

Figure 2 displays Eq. (30) for $\bar{\alpha}$ in the range of -2 to 6. Note that this solution represents the *entire classical frequency spectrum* for this problem. When Eq. (30) is plotted in the affine plane vs a_0/b_0 (for specific k_{x_0}, m, and n), the more familiar looking results of Fig. 3 present themselves.

Summary

Similarity rules have been formulated for a wide class of cross-ply symmetric plates with two simply supported parallel sides. Included in the formulation are the subcases of specially orthotropic and isotropic plates. The similarity rules have been illustrated by several simple examples and in one case a similarity *parameter* allows an even more compact solution. These rules permit one known solution to be transformed into an infinity of related solutions by a simple algebraic set of manipulations.

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